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THE LINEAR PROBLEM OF A VIBRATOR PERFORMING HARMONIC OSCILLATIONS AT SUPERCRITICAL FREQUENCIES IN A SUBSONIC BOUNDARY LAYER*

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The flow of a subsonic stream over a flat plate with a triangular vibrator fixed to it is studied. The vibrator begins to oscillate in the unperturbed boundary layer. The plate and vibrator are assumed to be heat insulated, and the vibrator dimensions are such that the flow can be defined by equations of the boundary layer with selfinduced pressure. The amplitude is assumed to be small, which enables these equations to be linearized. The Fourier and Laplace transformations respectively are used for the construction with respect to the longitudinal coordinate and time. Inverse transformations are investigated only for fixed values of the longitudinal coordinate and time approaching infinity. The amplitude of the pressure oscillations, which depends on the longitudinal coordinate, is obtained. When the vibrator oscillation frequency ω_0 is less than the critical value ω_{0*} the amplitude is damped both up- and downstream, when $\omega_0 = \omega_{0*}$ it is damped upstream and not damped downstream, and when $\omega_0 > \omega_*$ it is damped upstream and increases downstream. As the distance from the vibrator increases the perturbations degenerate into a Tollmien-Schlichting wave whose amplitude depends on the vibrator oscillation frequency.

Consider the flow over a heat-insulated body consisting of a plate at rest, a section of which contains a vibrator and oscillates, and the remaining section of which is at rest. Let the forward part be of length L^* and the rear part of length $O(L^*)$ (the asterisk superscript denotes dimensional quantities). Let the unperturbed stream be subsonic with a Mach number M_∞ less than unity by a finite quantity and with velocity V_∞^* directed along stationary parts of the body. The subscripts ∞ and w denote the parameters of gas in the steady unperturbed stream and on the body. We will use a Cartesian system of coordinates x and y with the origin at the junction point of the forward fixed part with the vibrator. We denote the time by t^*, the velocity vector components by v_x^* and v_y^*, the density by ρ^*, the pressure by p^*, the temperature by T^*, and the ratio of the specific heats by For simplicity we will assume the dependence of the first coefficient of viscosity on temperature to be locally linear (for $T^* \sim T_w^*$); $\lambda_1^*/\lambda_{1\infty}^* = CT'$, where $T' = T^*/T_\infty^*$, and the Prandtl number to be unity. Instead of the inverse value of the Reynolds number we will use the small parameter $\varepsilon = \mathrm{Re}_1^{-1/8}$ $(\mathrm{Re}_1 = \rho_\infty^* V_\infty^* L^*/\lambda_{1\infty}^*)$.

We select $O(L^*\varepsilon^3)$ for the longitudinal dimension of the vibrator, $O(L^*\varepsilon^5)$ for the oscillation amplitude, and $O(V_\infty^*/L^*\varepsilon^2)$ for the frequency. To define the motion produced by such a vibrator it is convenient to separate three characteristic regions /1, 2/: the upper or external region of the subsonic inviscid stream $(y^* = O(L^*\varepsilon^3))$, the middle region of the ordinary boundary layer $(y^* = O(L^*\varepsilon^4))$ and the lower region of the boundary layer with selfinduced pressure $(y^* = O(L^*\varepsilon^5))$. The difficulties of such schemes are basically related to the construction of a solution in the lower region. The flow parameters in the middle and upper regions can be written in explicit form /3, 4/.

Below, we shall deal only with the lower region. We introduce dimensionless dependent and independent variables indicated in /3, 4/, and use the notation described above for all quantities, except the velocity components, omitting the asterisk. The dimensionless longitudinal velocity will be denoted by u and the transverse velocity by v. By requiring that the conditions of merging with the conventional boundary layer should be satisfied as $x \to -\infty$ and $y \to \infty$, we obtain from the Navier-Stokes equations for the principal terms of the expansion as $\varepsilon \to 0$ a system of equations for the unsteady subsonic boundary layer with selfinduced pressure /3, 4/.

We specify the vibrator law of motion and its form for $t > 0$

$$y_w = \sigma f(t, x) = \sigma f_1(x) \sin \omega_0 t, \quad \sigma \ll 1, \quad \omega_0 > 0 \tag{1}$$

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where ω_0 is the dimensionless frequency, and function $f_1(x)$ defines a triangular form with parameters a and b ($f_1(x) = 0$ when $x \leqslant 0, 2x$ when $0 \leqslant x \leqslant b$, $2b(a - x)/(a - b)$ when $b \leqslant x \leqslant a$, and O when $x \geqslant a$). For the instants of time $t < 0$ we set $y_w = 0$, and assume that the boundary layer is unperturbed.

The smallness of the parameter σ enables us to expand the solution sought in series in powers of that parameter

$$u = y + \sigma u_1 + \ldots, \quad v = \sigma v_1 + \ldots, \quad p = \sigma p_1 + \ldots$$

Then the equations for the functions introduced will be linear

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad \frac{\partial p_1}{\partial y} = 0, \quad \frac{\partial u_1}{\partial t} + y\frac{\partial u_1}{\partial x} + v_1 = -\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2} \tag{2}$$

The conditions of interaction with the external stream give the connection between u_1 and p_1 as $y \to \infty$

$$u_1 \to \frac{1}{\pi}\int_{-\infty}^{x} dx_0 \int_{-\infty}^{\infty} \frac{p_1(t, x_1)}{x_1 - x_0} dx_1 \tag{3}$$

The conditions of adhesion to the body, if only principal terms are retained are

$$t > 0\colon\ u_1(t, x, 0) = -f_1(x)\sin\omega_0 t, \quad v_1(t, x, 0) = f_1(x)\,\omega_0\cos\omega_0 t \tag{4}$$

Since the oscillations begin in the unperturbed boundary layer, we have

$$u_1(0, x, y) = v_1(0, x, y) = p_1(0, x) = 0 \tag{5}$$

We shall seek a solution that has the following properties. For any finite $t > 0$ and $x \to \pm\infty$ the unknown functions approach zero, and the integrals of the aboslute values of these functions exist, and when x is finite and as $t \to \infty$ the unknown functions increase at a rate not greater than the exponential. The solution of problem (2)–(5) can then be sought by expanding it in a Fourier integral in the variable x and in a Laplace integral in the variable t

$$\bar{u}_1(\omega, k, y) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\int_{-\infty}^{\infty} u_1(t, x, y)\, e^{-\omega t - ikx} dx\, dt, \quad \mathrm{Re}\,\omega > l_0 > 0 \tag{6}$$

The problem of a vibrator oscillating for an infinitely long time in a subsonic boundary layer was considered in /5/. Becuase the motion investigated there was, for all x, already in a stable oscillation mode, the range of frequencies had to be limited $\omega_0 < \omega_{0*}$, where $\omega_{0*} = 2.298$ is the critical frequency predicted by the theory of stability. For supercritical frequencies $\omega_0 > \omega_{0*}$ in a formulation similar to that in /5/, a postulate was introduced in /6/ which stipulated the addition to the formal solution, which defines the oscillations of the whole plate limited with respect to amplitude, of a term that increases exponentially downstream. The postulate is based on the requirement of continuity of the solution on passing through the critical frequency, and on the experimental observation that no upstream propagation of strong perturbations occurs. There is no indication in the formulation (2)–(5) that a steady mode is reached in time for all x. That mode is reached for finite x as $t \to \infty$. This formulation enables the oscillations to be studied when $\omega_0 \geqslant \omega_{0*}$. It is not possible to use the Fourier transformation with respect to the variable t to derive the solution of (2)–(5), as was done in /5/, instead the Laplace transform (6) is required here.

Eliminating v_1 and p_1 from (2) and passing from u_1 to $\bar{u}_1$, we obtain

$$\frac{\partial^3 \bar{u}_1}{\partial y^3} = (\omega + iky)\frac{\partial \bar{u}_1}{\partial y}$$

The solution of this equation that satisfies the condition of boundedness of $\bar{u}_1$ as $y \to \infty$ has the form

$$\frac{\partial \bar{u}_1}{\partial y} = B(\omega, k)\,\mathrm{Ai}\,[(ik)^{1/3} y + \omega (ik)^{-2/3}]$$

where Ai is the Airy function /7/, $\arg i = \pi/2$, and $B(\omega, k)$ is an arbitrary function of its arguments. The limit condition (3) and the boundary condition (4) enable us to express $B(\omega, k)$ in terms of the function $\bar{f}(\omega, k)$ and determine $\bar{p}_1(\omega, k)$, where $\bar{f}$ and $\bar{p}_1$ are the functions $f(t, x)$ and $p_1(t, x)$ from (1) transformed by (5). We have

$$\bar{p}_1 = \omega_0 \mid k \mid \bar{f}_1(k)\,\mathrm{Ai}'(\Omega)/[(\omega_0^2 + \omega^2)\, Q(\Omega, k)]$$

$$\Omega = \omega (ik)^{-1/3}, \quad I_0 = \int_0^\infty \mathrm{Ai}(x)\,dx = \frac{1}{3}, \quad I_1(\Omega) = \int_0^\Omega \mathrm{Ai}(z)\,dz$$

$$Q(\Omega, k) = -\mathrm{Ai}'(\Omega) + (ik)^{1/3} \mid k \mid [I_0 - I_1(\Omega)]$$

$$\bar{f}_1(k) = -\sqrt{\frac{2}{\pi}}\frac{1}{k^2}\left(1 - \frac{a}{a-b}e^{-ikb} + \frac{b}{a-b}e^{-ika}\right)$$

where a prime denotes the derivative of the Airy function.

Let us calculate the pressure. An expression for p_1 is found by using the inverse Fourier and Laplace transforms

$$p_1 = -i2^{-3/2}\pi^{-1/2}(I_2 + I_3), \quad Q_2 = -Q\,|_{k<0}, \quad Q_2 = Q\,|_{k>0} \tag{7}$$

$$I_2 = \omega_0 \int_{-\infty}^{0} k\bar{f}_1(k)\, e^{ik}x\,dk \int_{l-i\infty}^{l+i\infty} \frac{\mathrm{Ai}'(\Omega)\, e^{\omega t}}{(\omega^2 + \omega_0^2)\, Q_2} |d\omega$$

$$I_3 = \omega_0 \int_{0}^{\infty} k\bar{f}_1(k)\, e^{ikx}\,dk \int_{l-i\infty}^{l+i\infty} \frac{\mathrm{Ai}'(\Omega)\, e^{\omega t}}{(\omega^2 + \omega_0^2)\, Q_3}\, d\omega$$

To separate the single-valued branches in the integrands in (7) we make a cut in the complex plane from the point O along the imaginary axis and select $\pi/2 > \arg k > -3\pi/2$.

The subdivision of the integral of k in (7) into I_2 and I_3 is connected with the fact that in I_2 as well as in I_3 the integrands are analytic functions. Equating to zero the expressions for Q_2 and Q_3 which appear in the denominators of I_2 and I_3, we obtain the dispersion relations for the subsonic boundary layer that were investigated in detail in /8/. The connection between ω and k defined by these relations differ substantially from the relation defined by the dispersion relation in the supersonic boundary layer /4, 9/. Thus, while for the external subsonic flow the dispersion relation has a root ω that passes from the left half-plane to the right half-plane, when k varies along the real axis, there are no such roots for the supersonic flow. It is precisely this root that determines the appearance of perturbations whose amplitude increases in the downstream direction.

When investigating p_1 in (7), we first consider the integral I_2. We separate its inner part, i.e. the integral of ω, denoting it by J_2, and investigate the roots of the denominator of integrand J_2 in the complex plane ω, as k varies along the negative part of the real axis. Using the results of /8/, we plot the trajectories of the first three roots of the dispersion equation $Q_2 = 0$ in Fig.1, denoting them by the numbers 1, 2, 3. All the remaining roots of the dispersion equation lie in the second quadrant.

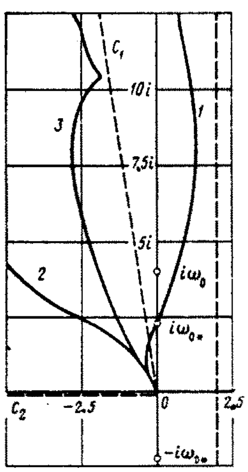

Fig.1

For all roots, beginning with the second, the inequality

$$\pi > \arg\, \omega_{2n}(k) > 0.556\pi,$$
$$n = 2, 3, \ldots$$

is satisfied.

Besides these roots, there are two more roots $i\omega_0$ and $-i\omega_0$ that are independent of k. The trajectory of the first root intersects the imaginary axis at the point $i\omega_{0*}$. The case of $\omega_0 > \omega_{0*}$ is shown in Fig.1. Let us transform the integral J_2. For this we deduct from and add to it the expression related to the first root of the dispersion equation $\omega_{21}(k)$. We have

$$J_2 = J_{20} + J_{21}, \quad J_{20} = \int_{l-i\infty}^{l+i\infty} \Phi_2 d\omega, \quad \Omega_{21} = \frac{\omega_{21}(k)}{(ik)^{1/3}} \tag{8}$$

$$\Phi_2 = \left\{\frac{\mathrm{Ai}'(\Omega)}{(\omega^2 + \omega_0^2)\, Q_2(\Omega, k)} - \frac{\mathrm{Ai}'(\Omega_{21})}{[\omega_{21}(k) + i\omega_0]\, Q_{2\omega}(\Omega_{21}, k)} \times \frac{1}{(\omega - i\omega_0)[\omega - \omega_{21}(k)]}\right\} e^{\omega t}$$

$$J_{21} = \frac{\mathrm{Ai}'(\Omega_{21})}{[\omega_{21}(k) + i\omega_0]\, Q_{2\omega}(\Omega_{21}, k)} \int_{l-i\infty}^{l+i\infty} \frac{e^{\omega t} d\omega}{(\omega - i\omega_0)[\omega - \omega_{21}(k)]}$$

$$Q_{2\omega} = \frac{\partial Q_2}{\partial \omega}\Big|_{\omega = \omega_{21}(k)}$$

This representation has the property that among the roots of the integrand denominator of Φ_2 there is no root $\omega = \omega_{21}(k)$. The remaining roots are the same as in J_2.

The integral J_{21} can be evaluated explicitly

$$J_{21} = 2\pi i \frac{\mathrm{Ai}'(\Omega_{21})}{[\omega_{21}(k) + i\omega_0]\, Q_{2\omega}(\Omega_{21}, k)} \left[\frac{e^{\omega_{21}(k)t}}{\omega_{21}(k) - i\omega_0} + \frac{e^{i\omega_0 t}}{i\omega_0 - \omega_{21}(k)}\right] \tag{9}$$

Let us transform the integral J_{20}. To do this we select, instead of the old integration path shown in Fig.1 by the vertical dash line, a path consisting of two rays C_1 and C_2 (Fig.1). Ray C_1 lies in the second quadrant and does not touch the pole trajectory, and ray C_2 coincides with the negative part of the real axis. Taking into account the residues at points $i\omega_0$ and $-i\omega_0$, we obtain

$$J_{20} = 2\pi i\,[\mathrm{res}\,\Phi_2(-i\omega_0) + \mathrm{res}\,\Phi_2(i\omega_0)] + \Big(\int_{C_2} + \int_{C_1}\Big)\Phi_2 d\omega \tag{10}$$

$$\mathrm{res}\,\Phi_2(-i\omega_0) = \frac{\mathrm{Ai}'(-\Omega_0)\,e^{-i\omega_0 t}}{-2i\omega_0 Q_2(-\Omega_0, k)}, \quad \Omega_0 = \frac{i^{1/3}\omega_0}{k^{2/3}}$$

$$\mathrm{res}\,\Phi_2(i\omega_0) = \left[\frac{\mathrm{Ai}'(\Omega_0)}{2i\omega_0 Q_2(\Omega_0, k)} + \frac{\mathrm{Ai}'(\Omega_{21})}{[\omega_0^2 + \omega_{21}^2(k)]\,Q_{2\omega}(\Omega_{21}, k)}\right] e^{i\omega_0 t}$$

Let us estimate the integrals along the rays C_2 and C_1 as $t \to \infty$. Their integrands have a form convenient for applying the Laplace lemma on asymptotic estimates of integrals, in accordance with which the basic contribution to the integration is made by the small neighbourhood of the point O. Then, using the boundedness of $|(I_0 - I_1(\Omega)/\mathrm{Ai}'(\Omega)|$ when ω varies along the integration path, and integrating with respect to k, we find that the contribution to the pressure p_1 is $O(t^{-6})$ as $t \to \infty$. Note that the estimate obtained is independent of x. Writing the integral of J_2 taken with respect to k as $t \to \infty$, we obtain

$$I_2 = 2\pi i\omega_0 \int_{-\infty}^{0} k\bar{f}_1(k)\,e^{ikx}\,[\mathrm{res}\,\Phi_2(-i\omega_0) + \mathrm{res}\,\Phi_2(i\omega_0)]\,dk + \omega_0 \int_{-\infty}^{0} k\bar{f}_1(k)\,e^{ikx} J_{21} dk + O(t^{-6}) \tag{11}$$

Formula (11) shows that for $t \gg 1$ the problem of determining the pressure for all values of x reduces to a single integral over k. Various problems may be considered here, for example, the determination of the velocity of motion of maximum amplitude, or problems of calculating the pressure along the whole of a straight line. Below, we consider the problem of reaching a stable oscillation mode for limited values of x. The need to introduce the additional quantity $\omega - i\omega_0$ in the denominator of the additional term in Φ_2 from (8) now becomes clear. Instead of $\omega - i\omega_0$ we can use for $\omega_0 \neq \omega_{0*}$ the simpler expression $\omega_{21}(k) - i\omega_0$. This cannot, however, be done for $\omega_0 = \omega_{0*}$, since for $k = k_{2*} = -1.0005$ the additional quantity would become infinite. Note that since when $\omega_0 \neq \omega_{0*}$ none of the terms in brackets in J_{21} in (9) and in $\mathrm{res}\,\Phi_2(i\omega_0)$ in (10) has a singularity along the integration path, before evaluating the integrals, the expressions proportional to $e^{i\omega_0 t}$ can be cancelled.

When $\omega = \omega_{0*}$ the expressions for J_{21} and $\mathrm{res}\,\Phi_2(i\omega_0)$ have no singularities at the point $k = k_*$, but among the two terms appearing in J_{21}, as well as in $\mathrm{res}\,\Phi_2(i\omega_0)$ each infinitely increases as $k \to k_*$. We bypass the point k_* along the arc of a small circle whose location is unimportant, since the point k_* is regular. But if the integration path does not contain the point k_*, the terms proportional to $e^{i\omega_0 t}$ can be cancelled along it. For the remaining terms we now obtain the rule of bypassing the point k_*: in both integrals in (11) the bypass must be carried out on one and the same side of the point k_*. We shall do so from below. In view of the above, the symbol $\mathrm{res}\,\Phi_2(i\omega_0)$ will be taken as containing only the first term shown in formula (10).

Let us consider the second integral in (11), and in view of the above remark

$$I_{21} = \int_{-\infty}^{0} \Phi_{21} dk, \quad \Phi_{21} = 2\pi i\omega_0 k\bar{f}_1(k)\,\frac{\mathrm{Ai}'(\Omega_{21})\,e^{ikx+\omega_{21}(k)t}}{Q_{2\omega}(\Omega_{21}, k)[\omega_{21}^2(k) + \omega_0^2]} \tag{12}$$

The roots $k_{21}(\omega_0)$ lying in the left half-plane of the equation $\omega_{21}^2(k) + \omega_0^2 = 0$ are shown in Fig.2 for ω_0 varying from 0 to ∞. When $\omega_0 = \omega_{0*}$ the equation has the root $k = k_{2*}$.

We pass now in the integral in (12) to the new integration path C_3. Computer calculations showed that the path C_3 may be selected below the trajectory $k = k_{21}(\omega_0)$ (Fig.2); then, for points $k \in C_3$ the following relations are satisfied: when $|k| \to \infty$ $\arg k \to -7\pi/8$, $\omega_{21} \to ik^2$, and $(\mathrm{Re}\,\omega_{21}(k))_{\max} = 0$ when $k = 0$, where the subscript max denotes the maximum value of the quantity.

As a result, we represent the integral (12) in the form

$$I_{21} = \int_{C_3} \Phi_{21} dk - 2\pi i\,\mathrm{res}\,\Phi_{21}(k_{21})\,\theta(\omega_0 - \omega_{0*}), \quad k_{21} = k_{21}(\omega_0) \tag{13}$$

$$\mathrm{res}\,\Phi_{21}(k_{21}) = -\pi k_{21}\bar{f}_1(k_{21})\,\mathrm{Ai}'(\Omega_{10}(k_{21}))\,[Q_{2k}(\Omega_{10}(k_{21}), k_{21})]^{-1} \times \exp(ik_{21}x + i\omega_0 t), \quad Q_{2k} = (\partial Q_2/\partial k)_\omega, \; \Omega_{10}(k_{21}) = i^{1/3}\omega_0 k_{21}^{-2/3}$$

where $\theta(\omega_0 - \omega_{0*})$ is the Heaviside function, and when $\omega_0 = \omega_{0*}$ there is no term containing $\mathrm{res}\,\Phi_{21}(k_{2*})$, since the point k_{2*} in the input integral (12) is bypassed from below. The

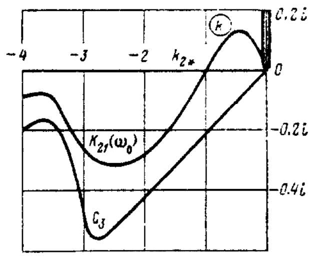

Fig.2

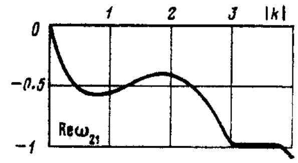

Fig.3

dependence of $\mathrm{Re}\,\omega_{21}(k)$ on $|k|$ is shown in Fig.3 for k varying along the path C_3. Since $\mathrm{Re}\,\omega_{21}(k)$ reaches its maximum when $k = 0$, the basic contribution to the integral along C_3 when $t \to \infty$ is provided by the neighbourhood of the point $k = 0$. Using the form of integrand as $k \to 0$, we obtain

$$\int_{C_3} \Phi_{21} dk = O(t^{-6}) \tag{14}$$

Note that the estimate (14) holds only for $x < t\,(\mathrm{Re}\,\omega_2(k)/\mathrm{Im}\,k)_{\max}$.

Let us now consider the integral I_3. For this we apply to the inner integral with respect to ω a transformation similar to that in (8), namely adding and subtracting the expression

$$\frac{\mathrm{Ai}'(\Omega_{31})}{[\omega_{31}(k) - i\omega_0]\, Q_{3\omega}(\Omega_{31}, k)} \frac{e^{\omega t}}{[\omega + i\omega_0][\omega - \omega_{31}(k)]}, \quad \Omega_{31} = \frac{\omega_{31}(k)}{(ik)^{2/3}}$$

where $\omega = \omega_{31}(k)$ is the first root of the dispersion equation $Q_3 = 0$. It can be shown that the trajectories of all roots of the equation $Q_3 = 0$ in the plane ω when k varies along the positive semiaxis, can be obtained from the trajectories of the roots $Q_2 = 0$, when k varies along the negative semiaxis, by symmetric reflection of the latter in the axis $\mathrm{Im}\,\omega = 0$. We can, then, write for I_3 a formula similar to (11) for the integral I_2. As the result we have

$$I_3 = 2\pi i\omega_0 \int_0^\infty k\bar{f}_1(k)\, e^{ikx} [\mathrm{res}\,\Phi_3(-i\omega_0) + \mathrm{res}\,\Phi_3(i\omega_0)]\, dk + \omega_0 \int_0^\infty k\bar{f}_1(k)\, e^{ikx} J_{31} dk + O(t^{-6}) \tag{15}$$

$$\mathrm{res}\,\Phi_3(-i\omega_0) = \left[\frac{\mathrm{Ai}'(-\Omega_0)}{-2i\omega_0 Q_3(-\Omega_0, k)} + \frac{\mathrm{Ai}'(\Omega_{31})}{[\omega_{31}^2(k) + \omega_0^2]\, Q_{3\omega}(\Omega_{31}, k)}\right] \times$$

$$e^{-i\omega_0 t}, \quad \mathrm{res}\,\Phi_3(i\omega_0) = \frac{\mathrm{Ai}'(\Omega_0)\, e^{i\omega_0 t}}{2i\omega_0 Q_3(\Omega_0, k)}, \quad \Omega_0 = \frac{i\omega_0}{(ik)^{2/3}}$$

$$J_{31} = 2\pi i \frac{\mathrm{Ai}'(\Omega_{31})}{[\omega_{21}^2(k) + \omega_0^2]\, Q_{3\omega}(\Omega_{31}, k)} (e^{\omega_{31}(k)t} - e^{-i\omega_0 t})$$

We cancel out in $\mathrm{res}\,\Phi_3(-i\omega_0)$ and J_{31} terms proportional to $e^{-i\omega_0 t}$, and when $\omega_0 = \omega_*$ we bypass the point $k_{3*} = 1.0005$ from below both in the first and in the second integral.

The symbol $\mathrm{res}\,\Phi_3(-i\omega_0)$ will be understood to contain only the first term given in formula (15). Passing in the second integral in (15) to integration along the ray C_4 lying below the trajectories $k = k_{31}(\omega_0)$, where $\omega_{31}(k_{31}) = -i\omega_0$, and has properties similar to those of path C_3, we obtain

$$I_{31} = \int_{C_4} \Phi_{31} dk - 2\pi i\, \mathrm{res}\,\Phi_{31}(k_{31})\, \theta(\omega_0 - \omega_{0*}), \quad k_{31} = k_{31}(\omega_0) \tag{16}$$

$$\mathrm{res}\,\Phi_{31}(k_{31}) = \pi k_{31}\bar{f}_1(k_{31})\, \mathrm{Ai}'(-\Omega_{10}(k_{31})) \times$$

$$[Q_{3k}(-\Omega_{10}(k_{31}), k_{31})]^{-1} \exp(ik_{31}x + i\omega_0 t)$$

$$Q_{3k} = (\partial Q_3/\partial k)_\omega, \quad \Omega_{10}(k_{31}) = i^{1/3}\omega_0 k_{31}^{-2/3}$$

There is not term with $\mathrm{res}\,\Phi_{31}(k_{3*})$ when $\omega_0 = \omega_{0*}$, since the bypassing of the point k_{3*} in (15) is carried out below it. Since the dependence of $\mathrm{Re}\,\omega_{21}(k)$ on $|k|$ when k varies along path C_4 is the same as the dependence $\mathrm{Re}\,\omega_{21}(k)$ along the ray C_3 (Fig.3) when k varies, if C_4 is selected to be symmetric to C_3 about the imaginary axis, then the main contribution as $t \to \infty$ is provided by the neighbourhood of the point $k = 0$. Using the expansion of the integrand of Φ_{31} as $k \to 0$, we obtain

$$\int_{C_4} \Phi_{31} dk = O(t^{-6}) \qquad (17)$$

Estimate (17) holds for $x < t\,(\mathrm{Re}\,\omega_{31}(k)/\mathrm{Im}\,k)_{\max}$. Let us determine the sum $I_2 + I_3$, and consequently in conformity with (7) also the pressure p_1 as $t \to \infty$, using (11) and (15) and also (13) and (16) and the estimates (14) and (17).

First, we collect the terms without integrals which appear only when $\omega_0 > \omega_{0*}$ the contribution of these terms to the pressure is denoted by p_{1r}

$$p_{1r} = -2^{-1/2}\pi^{1/2}\,(\mathrm{res}\,\Phi_{21}(k_{21}) + \mathrm{res}\,\Phi_{31}(k_{31}))$$

The analysis of the equations $\omega_{21}(k_{21}) = i\omega_0$ and $\omega_{31}(k) = -i\omega_0$ enables us to prove that

$$k_{31} = |k_{21}|\exp(-i\arg(k_{21}) - i\pi) = -(k_{21})_c$$

where the symbol $(\ldots)_c$ denotes the complex conjugate.

From the last formula we obtain the simple corollaries

$$-\Omega_{10}(k_{31}) = (\Omega_{10}(k_{21}))_c,\ Q_{3k}(-\Omega(k_{31}), k_{31}) = (Q_{2k}(\Omega_{10}k_{21}), k_{21})_c \qquad (18)$$
$$\mathrm{res}\,\Phi_{31}(k_{21}) = (\mathrm{res}\,\Phi_{21}(k_{21}))_c,\ \bar{f}_1(k_{31}) = (\bar{f}_1(k_{21}))_c$$

Then the expresion for p_1 takes the form

$$p_{1r} = -2^{1/2}\pi^{-1/2}\,\mathrm{Re}\,(\mathrm{res}\,\Phi_{21}(k_{21})) = \frac{1}{\pi}\,\mathrm{Im}\,(B_1(k_{21})\,e^{ik_{21}x})\cos\omega_0 t + \qquad (19)$$
$$\frac{1}{\pi}\,\mathrm{Re}\,(B_1(k_{21})\,e^{ik_{21}x})\sin\omega_0 t$$
$$B_1(k_{21}) = -3\pi e^{i\pi/3}k_{21}^{-4/3}\left(1 - \frac{a}{a-b}\,e^{-ik_{21}b} + \frac{b}{a-b}\,e^{-ik_{21}a}\right)\times$$
$$\mathrm{Ai}'(\Omega_{10}(k_{21}))\,[2\,(I_0 - I_1(\Omega_{10}(k_{21}))) + \Omega_{10}(k_{21})\,(1 - \omega_0/k_{21}^2)\,\mathrm{Ai}\,(\Omega_{10})(k_{21}))]^{-1}$$

We collect the integral terms in $I_2 + I_3$ that have no singularities in their integrands when $\omega_{0*} = \omega_{0r}$. Their contribution to the pressure is denoted by p_{1n}

$$p_{1n} = 2^{-1/2}\pi^{-1/2}\omega_0\left[\int_{-\infty}^{0} k\bar{f}_1(k)\,e^{ikx}\,\mathrm{res}\,\Phi_2(-i\omega_0)\,dk + \int_{0}^{\infty} k\bar{f}_1(k)\,e^{ikx}\,\mathrm{res}\,\Phi_3(i\omega_0)\,dk\right] \qquad (20)$$

In the first inegral in (20) we make the substitution $k = k_1 e^{-i\pi}$, and using transformations similar to transformations (18), obtain

$$\int_{-\infty}^{0} k\bar{f}_1(k)\,e^{ikx}\,\mathrm{res}\,\Phi_2(-i\omega_0)\,dk = \left(\int_{0}^{\infty} k\bar{f}_1(k)\,e^{ikx}\,\mathrm{res}\,\Phi_3(i\omega_0)\,dk\right)_c$$

As a result, the expression for p_{1n} takes the form

$$p_{1n} = 2^{1/2}\pi^{-1/2}\omega_0\,\mathrm{Re}\left(\int_{0}^{\infty} k\bar{f}_1(k)\,e^{ikx}\,\mathrm{res}\,\Phi_3(i\omega_0)\,dk\right) = \qquad (21)$$
$$\frac{1}{\pi}\,\mathrm{Im}\left(\int_{0}^{\infty}\Phi_n\,dk\right)\cos\omega_0 t + \frac{1}{\pi}\,\mathrm{Re}\left(\int_{0}^{\infty}\Phi_n\,dk\right)\sin\omega_0 t$$
$$\Phi_n = -\frac{e^{ikx}}{k}\left(1 - \frac{a}{a-b}\,e^{-ikb} + \frac{b}{a-b}\,e^{-ika}\right)\mathrm{Ai}'(\Omega_0)/Q_3(\Omega_0, k)$$

We now collect the integral terms in $I_2 + I_3$ with singularities in their integrands when $\omega_0 = \omega_{0*}$. Their contribution to the pressure is

$$p_{1s} = 2^{-1/2}\pi^{-1/2}\omega_0\left[\int_{-\infty}^{0} k\bar{f}_1(k)\,e^{ikx}\,\mathrm{res}\,\Phi_2(i\omega_0)\,dk + \int_{0}^{\infty} k\bar{f}_1(k)\,e^{ikx}\,\mathrm{res}\,\Phi_3(-i\omega_0)\,dk\right] \qquad (22)$$

The bypassing of the singular points $k = \pm 1.0005$ when $\omega_0 = \omega_{0*}$ occurs from below in the first and second integral. Making the change of variables $k = k_1\exp(-2i\arg k_1 - i\pi)$ in the second integral in (22) and using transformations similar to (18), we obtain

$$p_{1s} = 2^{1/2}\pi^{-1/2}\omega_0\,\mathrm{Re}\left(\int_{-\infty}^{0} k\bar{f}_1(k)\,e^{ikx}\,\mathrm{res}\,\Phi_2(i\omega_0)\,dk\right) = \qquad (23)$$
$$\frac{1}{\pi}\,\mathrm{Im}\left(\int_{-\infty}^{\infty}\Phi_s\,dk\right)\cos\omega_0 t + \frac{1}{\pi}\,\mathrm{Re}\left(\int_{-\infty}^{0}\Phi_s\,dk\right)\sin\omega_0 t$$
$$\Phi_s = -\frac{e^{ikx}}{k}\left(1 - \frac{a}{a-b}\,e^{-ikb} + \frac{b}{a-b}\,e^{-ika}\right)\mathrm{Ai}'(\Omega_0)/Q_2(\Omega_0, k)$$

Collecting the results obtained in (19), (21), and (23), we write the expression for the pressure as $t \to \infty$

$$p_1 = p_{1n} + p_{1s} + p_{1r}\,\theta\,(\omega_0 - \omega_{0*})$$

Thus the pressure can be written in the form

$$p_1 = p_{1n} + p_{1s} + p_{1r}\theta(\omega_0 - \omega_{0*}) = \frac{1}{\pi}\operatorname{Im}(\Phi_p)\cos\omega_0 t + \frac{1}{\pi}\operatorname{Re}(\Phi_p)\sin\omega_0 t, \quad \Phi_p = \int_{-\infty}^{\infty}\Phi dk + B_1(k_{21})e^{ik_{21}x}\theta(\omega_0 - \omega_{0*}) \tag{24}$$

$$\Phi = -\frac{e^{ikx}}{|k|}\left(1 - \frac{a}{a-b}e^{-ikb} - \frac{b}{a-b}e^{-ika}\right)\operatorname{Ai}'(\Omega_0)/Q(\Omega_0, k)$$

The quantity Q appearing in the expression for Φ is connected with Q_2 and Q_3 by (7). Note that expression (24) for the pressure when $\omega_0 < \omega_{0*}$ is the same as the expression for the pressure appearing in /5/ (of course, taking into account that the time t_5 in /5/ is related to the time t by the formula $t = t_5 + \pi/2\omega_0$). The quantity p_{1r} should not be taken into account, when $\omega_0 = \omega_{0*}$, however, since the integral Φ_p bypasses the point $k = k_{2*}$ from below, then according to /5/ the following representation holds:

$$x \gg 1, \quad \Phi_p = B_1(k_{2*})e^{ik_{2*}x} + O\left(\frac{1}{x^2}\right); \quad x \ll -1, \quad \Phi_p = O\left(\frac{1}{x^2}\right) \tag{25}$$

When $\omega_0 > \omega_{0*}$ the pole of the function Φ passes into the lower half-plane /5/, then for large values of $|x|$, for the integral of Φ the representation

$$x \gg 1, \quad \int_{-\infty}^{\infty}\Phi dk = O\left(\frac{1}{x^2}\right); \quad x \ll -1, \quad \int_{-\infty}^{\infty}\Phi dk = -B_1(k_{21})e^{ik_{21}x} + O\left(\frac{1}{x^2}\right)$$

holds.

Now, taking into account the expression for p_{1r}, we obtain for Φ_p

$$x \gg 1, \quad \Phi_p = B_1(k_{21})e^{ik_{21}x} + O\left(\frac{1}{x^2}\right); \quad x \ll -1, \quad \Phi_p = O\left(\frac{1}{x^2}\right) \tag{26}$$

The calculation of the pressure p_1 based on formula (24) is described in /5/. Curves of the pressure for frequencies $\omega_0 = 2$, $\omega_0 = \omega_{0*} = 2.298$, and $\omega_0 = 2.5$ for instants of time $t = 2\pi(N + 1/4)/\omega_0$ and vibrator parameters $a = 2$, $b = 1$ are shown in Fig.4. It follows from formulas (25) and (26) and these curves that perturbations propagate upstream only insignificantly. The perturbations that drift downstream with subcritical frequencies decrease as x increases, when the frequency is critical, their amplitude is independent of x, and is determined by the coefficient $B_1(k_{2*})$, and when the frequencies are supercritical, the perturbations increase exponentially as given by (26). We note in conclusion that the postulate introduced in /6/ is in complete agreement with (24).

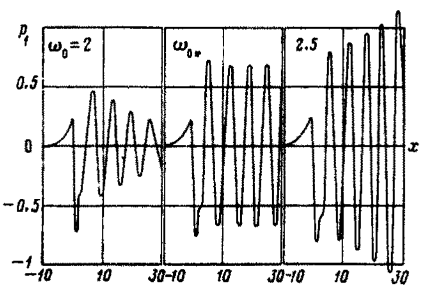

Fig.4

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THE EFFECTIVE THERMAL CONDUCTIVITY OF A SUSPENSION*

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The effective thermal conductivity of an inhomogeneous suspension is considered for the case of low and moderate volume densities of randomly distributed spherical particles. A mathematical apparatus of convolutions of the Λ-functions is developed enabling closed formulas to be derived for the dipole moment of a particle in the system. An exact expression for the dipole moment averaged over the ensemble that is accurate to terms of the order of the square of the particle density is given for a spatially homogeneous distribution of particles. The effective thermal conductivity of the suspension is calculated to the same approximation. It is shown that when the region occupied by the spherical particles represents an ellipsoid of revolution and the temperature gradient away from this region tends to a given constant value, the effective thermal conductivity becomes independent of the ratio of the ellipsoid semiaxes, i.e. independent of the form of the region occupied by the system.

The effective thermal conductivity of a homogeneous suspension was studied earlier in /1—7/. Maxwell calculated the effective electrical conductivity of a mixture to terms of the order of the volume concentration of the spherical inclusions. The effective thermal conductivity is easily calculated to the same approximation, since the problems of determining the thermal and electrical conductivity are mathematically equivalent. The same problem is encountered in computing the dielectric permeability of two-phase mixtures /8/ and in determining the effective shear modulus of a homogeneous and isotropic composite material /9, 10/.

A cell model was used in /2—5/ to compute the effective thermal and electrical conductivity of suspensions at moderate and high particle densities. It was assumed that the particle was situated at the centre of a spherical cell, and the medium outside it possessed the required effective thermal conductivity. The drawback of this method lies in the arbitrariness of the choice of the cell boundary. A method of calculating the effective thermal conductivity of the media with spherical inclusions situated at the nodes of various types of cubic lattices at moderate particle densities was given in /6/, where a review of the earlier investigations concerned with computing the thermal conductivity in analogous media at low volume densities was also given. The effective thermal conductivity of a homogeneous suspension with randomly distributed particles was calculated to terms of the order of the square of the particle density in /7/, using the method given earlier in /11/.

1. Formulation of the problem. Let a region of volume V containing N identical spherical particles of constant thermal conductivity $\varkappa' \neq \varkappa$ exist in an infinite medium filled with a material of constant thermal conductivity $\varkappa$. We assume that away from V a steady temperature distribution is given with constant gradient $\mathbf{k}$. The temperature field T will depend, at any point r, on the position of the particle centres determined by the radius vectors $\mathbf{r}_1, \ldots, \mathbf{r}_N$. We shall denote the complete set of these radius vectors by R_N. We will introduce an unconditional correlation function $f_N(R_N)$ such that

$$\frac{1}{V^N} f_N(R_N)\, dR_N$$

denotes the probability of finding the particle centres, respectively, within the small volumes $d^3r_1, \ldots, d^3r_N$ beside the points $\mathbf{r}_1, \ldots, \mathbf{r}_N$. We introduce the conditional correlation function $f_{N-1}(R_{N-1}; \mathbf{r}_N)$ defined in such a manner that

$$\frac{1}{V^{N-1}} f_{N-1}(R_{N-1}; \mathbf{r}_N)\, dR_{N-1}$$

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